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Chapter 1 part-1: 2D Potential Flow

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The General Eqns of Fluid Dynamics

Recall that:

The General Eqns of Fluid Dynamics

… were developed based on **conservation laws**:

- mass conservation:

  → ***continuity eqn.***

- momentum conservation /Newton's 2nd Law:

  → ***momentum eqn.***

- angular momentum conservation

  → **angular *momentum eqn.***

- energy conservation /1st Law of Thermo:

  → ***energy eqn.***

Recall the
Two Approaches used to Develop the Eqns.

1. Integral Approach
   – in which the size of the **CV is finite**

2. Differential Approach
   – in which the size of the **CV** is **infinitesimally small**

Recall the
Two Approaches used to Develop the Eqns…

- the integral approach is used when we seek an estimate of 'gross' effects over a finite flow region (e.g., estimates of mass flow, induced force, energy/power involved).

whereas, …

Recall the
Two Approaches used to Develop the Eqns…

….

- The differential approach is used when we seek the point-by-point details of a flow pattern by analyzing an infinitesimal region of a flow, i.e., when we seek ‘local’ values

General

Integral Governing Eqns

General Continuity Eqn. - integral form

$$\int_{\mathrm{CV}} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \int_{\mathrm{CS}} \rho(\mathbf{V} \cdot \mathbf{n})\, dA = 0$$

General Linear Momentum Eqn - integral form

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})\, dA$$

where

V is fluid velocity relative to an inertial (non-accelerating coordinate system,

Σ**F** is the **vector** sum of all forces acting on the CV material considered as a free body

$\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{body forces}} + \Sigma \mathbf{F}_{\text{surface forces}}$

General ……… Eqn - integral form

etc …

Please refer to e.g., Frank M. White, for the remaining integral eqns

General

Differential Governing Eqns.

kinematics aspect of a flow (in Cartesian co-ord.)

- **The velocity field:**

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{i}u(x, y, z, t) + \mathbf{j}v(x, y, z, t) + \mathbf{k}w(x, y, z, t)$$

- **The acceleration field**

x-component

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

i.e.,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})u$$

- **The total acc. field**

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}$$

The total time derivative may also be applied to any other dependent variable; e.g., the pressure:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\,\frac{\partial p}{\partial x} + v\,\frac{\partial p}{\partial y} + w\,\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V}\cdot\nabla)p$$

N.B.

- Wherever convective effects occur, the differential eqns. become non-linear, thus making them too complicated to solve them.

General
Differential Continuity Eqn

General Continuity Eqn. – diff approach

(Ref., Frank W. White)

- Consider:

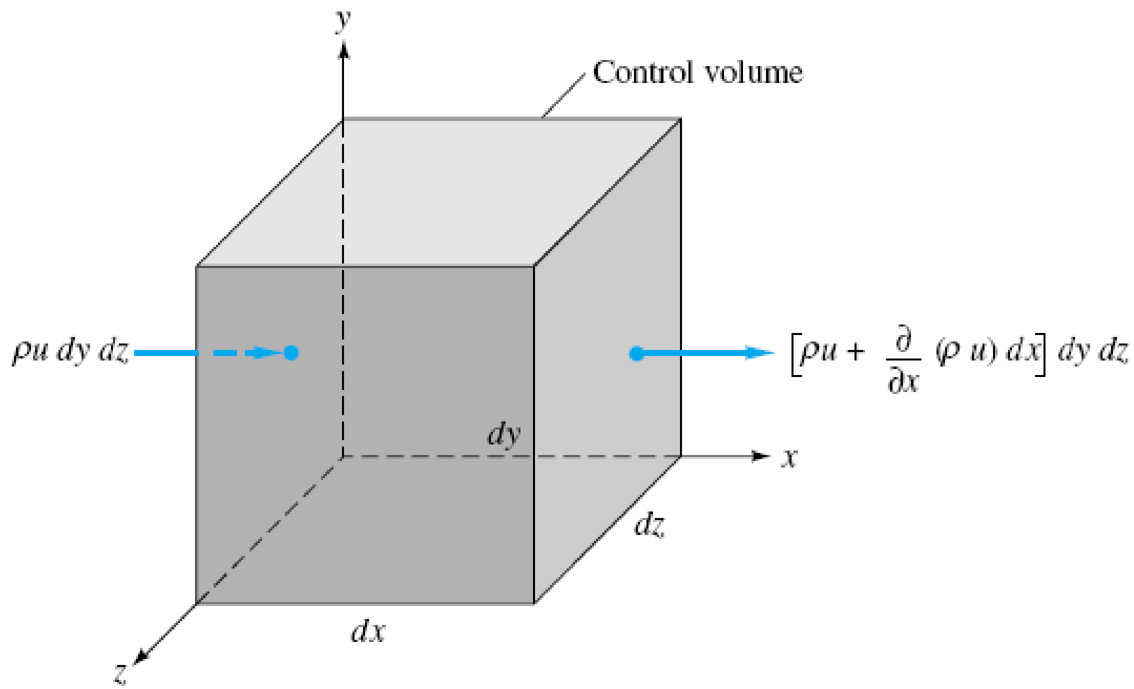


Fig. 4.1 Elemental cartesian fixed control volume showing the inlet and outlet mass flows on the x faces.

General Continuity Eqn. – diff approach…

- Calculating the **net mass flux across all the six faces** of the fixed diff. CV, and

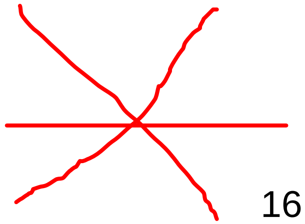
- applying mass conservation principle, we get the following continuity eqn.:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

- It is named 'continuity' eqn. because there is no underlying assumption except the continuum concept.

- The same eqn. can be written **in compact form** as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$



General Continuity Eqn. – diff approach…

Alternatively,

- Integral continuity eqn. written for finite CV: $\Rightarrow$ $$\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \int_{CS} \rho(\mathbf{V}\cdot\mathbf{n})\, dA = 0$$
- Applying Gauss Divergence Thm to the convective term: $\Rightarrow$ $$\int_{cs} \rho(\vec{V}\bullet\vec{n})dA = \int_{cv} div(\rho\vec{V})dv$$
- Substitute this for the convective term $\Rightarrow$ $$\int_{cv} \frac{\partial \rho}{\partial t} dv + \int_{cv} div(\rho\vec{V})dv = 0$$
- Let the size of the finite control volume approach infinitesimal, $\Rightarrow$ $$\frac{\partial \rho}{\partial t} + div(\rho\vec{V}) = 0$$
- which is the same as the previous expression $\Rightarrow$ $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0$$

General Continuity Eqn. – diff approach…

- Please refer to, e.g., Frank M. White for the rest of the eqns, including when other coordinate systems are used to describe the flow.

General Linear Momentum Eqn. – differential form

General Linear Momentum Eqn.. – diff. form

- Consider:

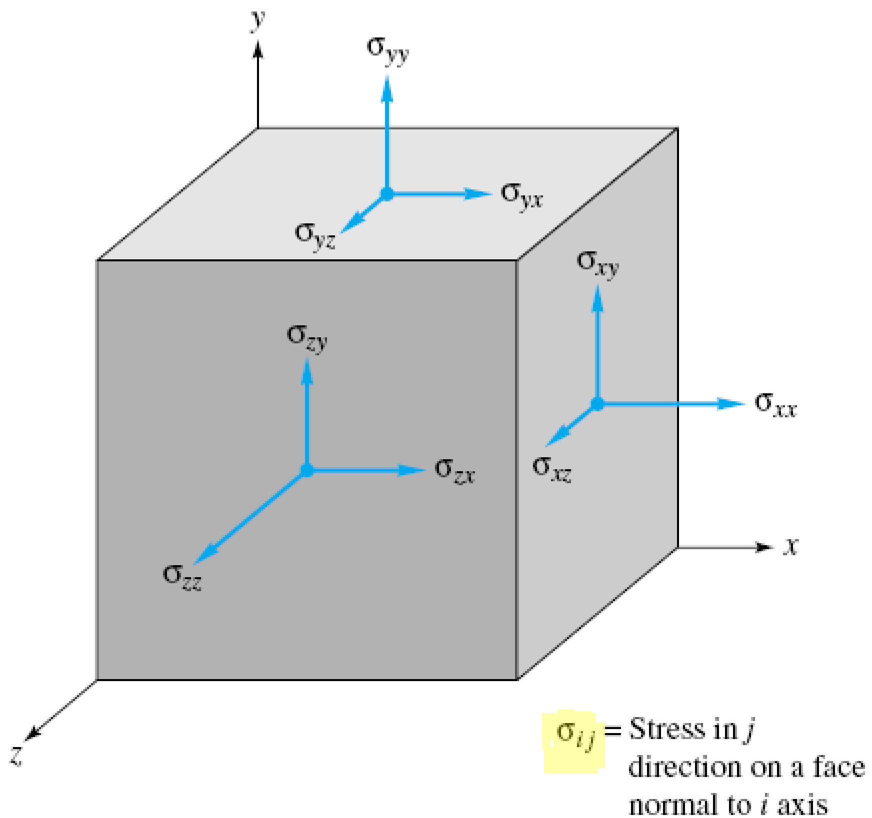


Fig. 4.3 Notation for stresses.

General Linear Momentum Eqn.. – diff. form

- and this fig.:-

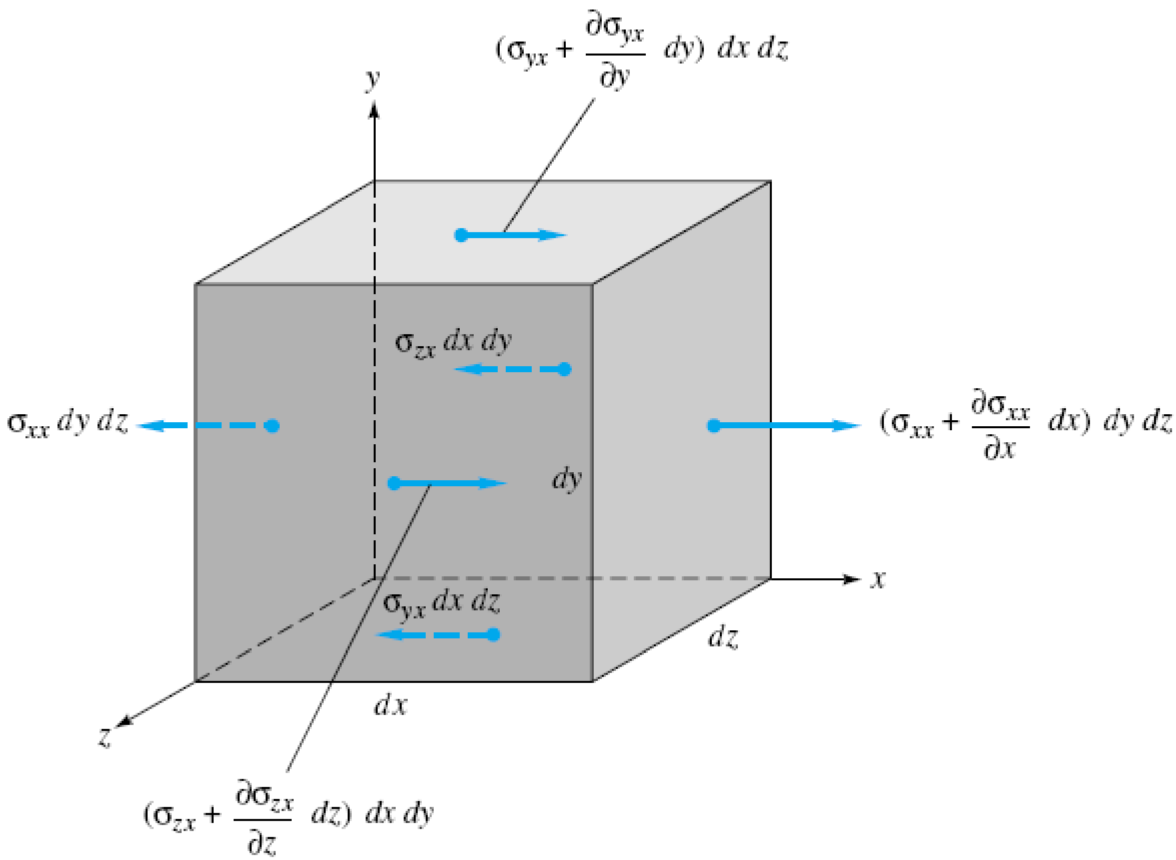


Fig. 4.4 Elemental cartesian fixed control volume showing the surface forces in the x direction only.

General Linear Momentum Eqn.– diff. form

- Ref. to the fixed CV, with inlet and outlet momentum fluxes as shown in the last slides, and remembering Newton's 2nd Law:
  - Net force = rate of change of linear momentum of the system instantly located in the CV, i.e.,

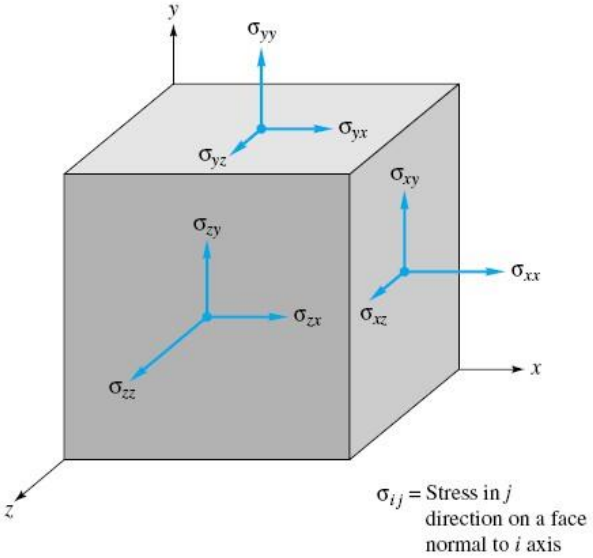
$$\sum \mathbf{F} = dx\, dy\, dz \left[\frac{\partial}{\partial t}(\rho \mathbf{V}) + \frac{\partial}{\partial x}(\rho u \mathbf{V}) + \frac{\partial}{\partial y}(\rho v \mathbf{V}) + \frac{\partial}{\partial z}(\rho w \mathbf{V}) \right]$$

General Linear Momentum Eqn.. – diff. form …

$\sum$ F : is the sum of **body force** and **surface forces**.

- the ***body force***,

  $d\mathbf{F}_{grav} = \rho \mathbf{g}\,(dx\,dy\,dz)$



- In **general**, the **surface forces** are due to ***pressure*** (p) and due to ***viscous stresses*** (τ_{ij}) which arise from motion with velocity gradients on the sides of the CV.

General Linear Momentum Eqn.. – diff. form …

- substituting for $\Sigma\mathbf{F}$ in the Newton's eqn, we obtain the basic momentum eqn. for a general flow., in compact form:

$$\rho\mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} = \rho \frac{d\mathbf{V}}{dt}$$

where,

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} + w \frac{\partial \mathbf{V}}{\partial z}$$

- In words, the vector eqn. says:-

Gravity force/unit volume **-** pressure force/unit volume **+** viscous force/unit volume **=** density x acc.,i.e., inertia force/unit volume.

General Linear Momentum Eqn.. – diff. form …

- The three components of the momentum eqn. are:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

- Note the mathematical complexity of the eqn. It is valid for any fluid undergoing through any type of motion.
- The last three “convective” terms on the RHS of each component eqn. are non-linear, and this complicates the mathematical analysis.

Linear momentum eqn. ...

- For incompressible flow of Newtonian fluids, the viscous stresses are approximated by:

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

General Linear Momentum Eqn.. – diff. form …

- Substituting the preceding expressions of viscous stresses into the general momentum eqn., we obtain the following reduced form, valid for an *incompressible* flow of *Newtonian* fluid with *constant viscosity*:
- These are named **Navier-Stokes eqn**. (2nd order, non-linear)

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\frac{dw}{dt}$$

General ……. Eqn.. – diff. form …

etc …

Please refer to e.g., Frank M. White, for the remaining idifferential eqns

Summary of the General Forms of Differential Governing Eqns.

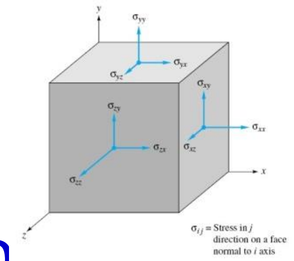
Continuity:	$\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Momentum:	$\rho \dfrac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$
Energy:	$\rho \dfrac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k\, \nabla T) + \Phi$

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$



- They contain the unknowns: ρ, $\mathbf{V}$, p, T
- and so - one additional relation is needed to close the system of eqns.:

$$\rho = \frac{p}{RT}$$

- Plus the necessary **boundary conditions**!!!

Summary of the Reduced Forms of Diff Eqns. for 2-D Potential Flow

Continuity ……….

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla \cdot (\cancel{\rho} \mathbf{V}) = 0$$

Linear Momentum …

(Euler's eqn.)

$$\cancel{\rho g} - \nabla p = \rho \frac{dV}{dt}$$